1. Consider the function $f(x, y) = \sin(x^2 - y^2)$.

a) (2 points) Find the gradient ∇f.

$$\nabla f = (f_x, f_y) = \boxed{(2x\cos(x^2-y^2),\ -2y\cos(x^2-y^2))}$$

b) (3 points) At the point $(x, y) = (1, 1)$, find the direction in which the directional derivative is maximum.

The direction of the maximal directional derivative is given by $\nabla f(1,1) = \boxed{(2,-2)}$

c) (2 points) At the point $(x, y) = (1, 1)$, find the direction in which the directional derivative is zero.

Take $\vec{u} = (a, b)$ to be the unit vector in this direction.

$D_{\vec{u}} f(1,1) = \nabla f(1,1) \cdot \vec{u} = (2,-2)\cdot(a,b) = 2a - 2b$.

$D_{\vec{u}} f(1,1) = 0 \Rightarrow 2a - 2b = 0 \Rightarrow a = b$.

$\Rightarrow \vec{u}$ is the unit vector with $a = b$

$$\Rightarrow \vec{u} = \pm \frac{(1,1)}{|(1,1)|} = \boxed{\pm \frac{1}{\sqrt{2}}(1,1)}$$

d) (3 points) Evaluate the limit

very tricky!

$$\lim_{\delta\to 0} \frac{f(1+\delta, 1-2\delta)}{\delta}.$$

Set $\vec{r}(t) = (1+t, 1-2t) \Rightarrow f(\vec{r}(0)) = f(1,1) = \sin(1^2-1^2) = 0$

$$\lim_{\delta\to 0} \frac{f(1+\delta, 1-2\delta)}{\delta} = \lim_{\delta\to 0} \frac{f(\vec{r}(\delta)) - f(\vec{r}(0))}{\delta}$$

$$= \left.\frac{df}{dt}\right|_{t=0} = \nabla f(\vec{r}(0))\cdot \vec{r}\,'(0)$$

chain rule

$$= \nabla f(1,1)\cdot(1,-2)$$

$$= (2,-2)\cdot(1,-2) = \boxed{6}$$

Note (1) There is a conceptual way to answer part (d) using directional derivatives.

$f(1,1) = \sin(1^2 - 1^2) = 0$

$$\Rightarrow \lim_{\delta \to 0} \frac{f(1+\delta, 1-2\delta)}{\delta} = \lim_{\delta \to 0} \frac{f(1+\delta, 1-2\delta) - f(1,1)}{\delta}$$

This is the change rate of f at $(1,1)$ when you move by the vector $\vec{v} = (1,-2)$ every second.

Take $\vec{u}$ to be the unit vector of $\vec{v}$.

Then the above limit is the change rate of f at $(1,1)$ in the direction of $\vec{u}$ and speed $|\vec{v}|$.

So we compute

$$|\vec{v}| D_{\vec{u}} f(1,1) = |\vec{v}| \nabla f(1,1) \cdot \vec{u} = |\vec{v}| \nabla f(1,1) \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$= \nabla f(1,1) \cdot \vec{v} = (2,-2) \cdot (1,-2) = 6$$

(2) You can also use L'Hopital's rule.

$$\lim_{\delta \to 0} f(1+\delta, 1-2\delta) = f(1,1) = \sin(1^2 - 1^2) = 0$$

$$\Rightarrow \lim_{\delta \to 0} \frac{f(1+\delta, 1-2\delta)}{\delta} \underset{\text{L'Hopital}}{=} \left.\frac{df}{dt}\right|_{t=0} \underset{\text{chain rule}}{=} \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$

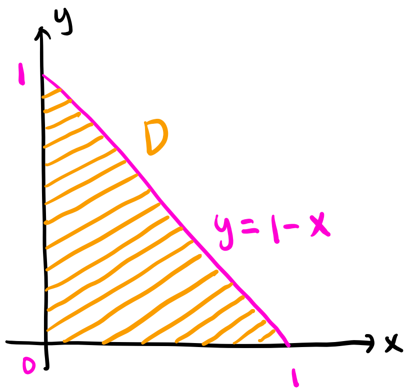
$$= \nabla f(1,1) \cdot (1,-2)$$

$$= (2,-2) \cdot (1,-2) = 6$$

2. Consider the double integral

$$\int\int_D x^2y^2\,dA.$$

a) (6 points) Evaluate the integral if D is the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$.



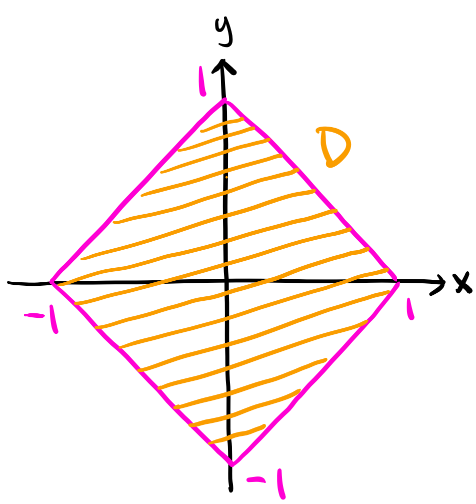
$D: 0 \le x \le 1,\ 0 \le y \le 1-x$

$$\iint_D x^2y^2\,dA = \int_0^1\int_0^{1-x} x^2y^2\,dy\,dx = \int_0^1 \left.\frac{x^2y^3}{3}\right|_{y=0}^{y=1-x} dx$$

$$= \frac{1}{3}\int_0^1 x^2(1-x)^3\,dx = \frac{1}{3}\int_0^1 x^2 - 3x^3 + 3x^4 - x^5\,dx$$

$$= \frac{1}{3}\left(\frac{x^3}{3} - \frac{3}{4}x^4 + \frac{3}{5}x^5 - \frac{x^6}{6}\right)\Bigg|_{x=0}^{x=1} = \boxed{\frac{1}{180}}$$

b) (4 points) Evaluate the integral if D is the square with vertices at $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$.



Divide D into four parts D_1, D_2, D_3, D_4 in each quadrant.

x^2y^2 is even with respect to x and y

$$\Rightarrow \iint_D x^2y^2\,dA = 4\iint_{D_1} x^2y^2\,dA$$

$$= \underset{\substack{\uparrow \\ \text{part (a)}}}{4\cdot\frac{1}{180}} = \boxed{\frac{1}{45}}$$

3. Consider the solid sphere $0 \le x^2 + y^2 + z^2 \le 1$. Suppose the density at a point in the solid is equal to $1/r$, where $r = \sqrt{x^2 + y^2 + z^2}$ (distance from the origin).

a) (6 points) Find the total mass of the solid.

E : the solid sphere $x^2+y^2+z^2 \le 1$

$\Rightarrow 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi,\ 0 \le \rho \le 1$ in spherical coordinates.

The density is $\rho(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

$$\Rightarrow m = \iiint_E \rho(x,y,z)\,dV = \iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV$$

Jacobian

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \frac{1}{\rho}\cdot \underline{\rho^2 \sin\varphi}\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\rho^2}{2}\sin\varphi\,\Big|_{\rho=0}^{\rho=1} d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{2}\sin\varphi\,d\varphi\,d\theta = \int_0^{2\pi} -\frac{1}{2}\cos\varphi\,\Big|_{\varphi=0}^{\varphi=\pi} d\theta$$

$$= \int_0^{2\pi} 1\,d\theta = \boxed{2\pi}$$

b) (4 points) Find $a > 0$ such that the mass inside the volume $0 \le x^2 + y^2 + z^2 \le a^2$ is equal to the mass outside.

E : the solid sphere $x^2+y^2+z^2 \le a^2$

$\Rightarrow 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi,\ 0 \le \rho \le a$ in spherical coordinates.

$$\Rightarrow m = \iiint_E \rho(x,y,z)\,dV = \iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV$$

Jacobian

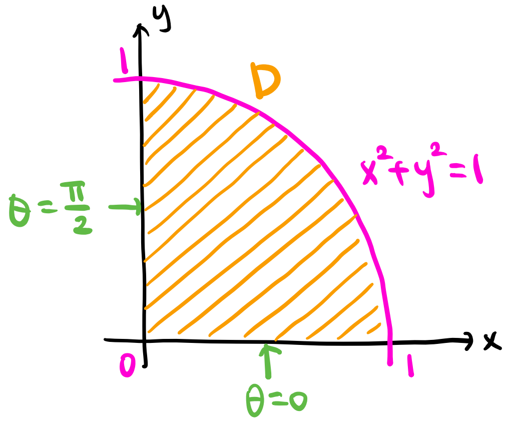
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \frac{1}{\rho}\cdot \underline{\rho^2 \sin\varphi}\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\rho^2}{2}\sin\varphi\,\Big|_{\rho=0}^{\rho=a} d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{a^2}{2}\sin\varphi\,d\varphi\,d\theta = a^2\int_0^{2\pi}\int_0^{\pi} \frac{1}{2}\sin\varphi\,d\varphi\,d\theta \underset{(a)}{=} 2a^2\pi$$

We want $m \underset{(a)}{=} \frac{1}{2}\cdot 2\pi \Rightarrow 2a^2\pi = \pi \Rightarrow a = \boxed{\frac{1}{\sqrt{2}}}$

4. Consider the quarter disc $0 \le x^2 + y^2 \le 1$, $x \ge 0$, $y \ge 0$. Suppose the density (mass per unit area) at a point in the quarter disc is given by $\rho(x, y) = x^2 + y^2$.

a) (3 points) Find the total mass of the quarter disc.



In polar coordinates, the region D is given by $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 1$.

Jacobian

$$m = \iint_D \rho(x,y)\,dA = \iint_D x^2+y^2\,dA = \int_0^{\pi/2}\int_0^1 r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left.\frac{r^4}{4}\right|_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\,d\theta = \boxed{\frac{\pi}{8}}$$

b) (4 points) Find $\bar{x}$, the x-coordinate of the center of mass.

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{8}{\pi}\iint_D x(x^2+y^2)\,dA$$

Jacobian

$$= \frac{8}{\pi}\int_0^{\pi/2}\int_0^1 r\cos\theta \cdot r^2 \cdot r\,dr\,d\theta = \frac{8}{\pi}\int_0^{\pi/2} \frac{r^5}{5}\cos\theta\,d\theta \Big|_{r=0}^{r=1} d\theta$$

$$= \frac{8}{\pi}\int_0^{\pi/2} \frac{1}{5}\cos\theta\,d\theta = \frac{8}{\pi}\cdot\frac{1}{5}\sin\theta \Big|_{\theta=0}^{\theta=\pi/2} = \boxed{\frac{8}{5\pi}}$$

c) (3 points) Find the distance of the center of mass from the origin.

The region D remains unchanged upon swapping x and y

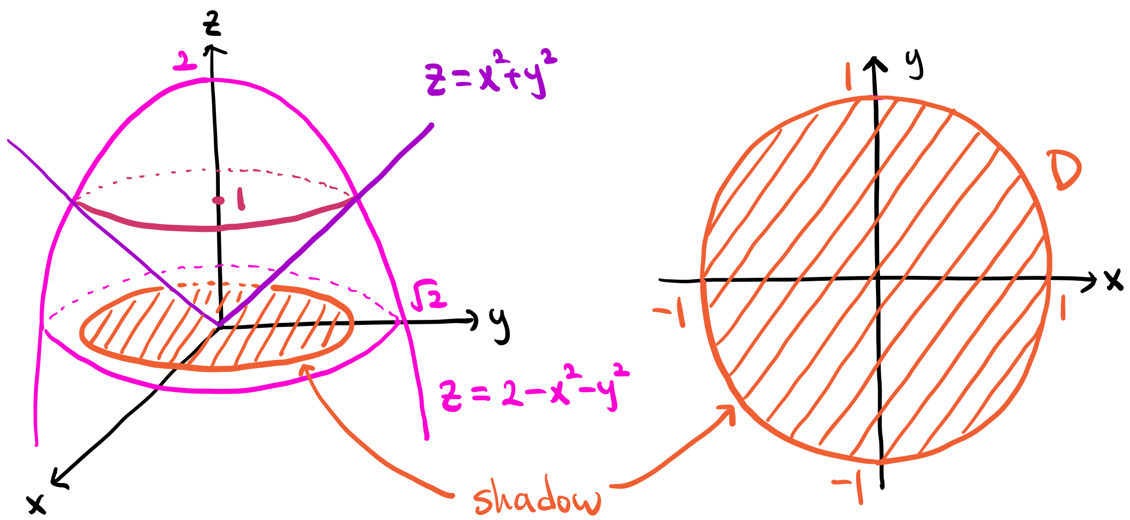
$$\Rightarrow \bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{1}{m}\iint_D y(x^2+y^2)\,dA$$

$$\underset{\text{swapping } x \text{ and } y.}{=} \frac{1}{m}\iint_D x(x^2+y^2)\,dA \underset{(b)}{=} \bar{x} = \frac{8}{5\pi}$$

The distance is $\sqrt{\bar{x}^2+\bar{y}^2} = \sqrt{2}\,\bar{x} = \boxed{\frac{8\sqrt{2}}{5\pi}}$

5. Consider the paraboloid surface $z = 2 - (x^2 + y^2)$ and the conical surface $z^2 = x^2 + y^2$ with $z \geq 0$.

a) (3 points) Find the value of z at which the two surfaces intersect.



The intersection is given by $z = 2 - x^2 - y^2$ and $z^2 = x^2 + y^2$

$\Rightarrow z = 2 - z^2 \Rightarrow z^2 + z - 2 = 0 \underset{z \geq 0}{\Rightarrow} z = \boxed{1}$

b) (7 points) Find the surface area of the part of the paraboloid above that value of z.

The paraboloid is the graph of $f(x,y) = 2 - x^2 - y^2$.

The intersection circle is given by $z = 1$, $x^2 + y^2 = 1$.

$\Rightarrow$ The shadow D of the surface is given by $x^2 + y^2 \leq 1$.

$\Rightarrow$ In polar coordinates, D is given by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$.

$$\text{Area} = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA$$

Jacobian

$$= \int_0^{2\pi} \int_0^1 \sqrt{1+4r^2} \cdot r\, dr\, d\theta \underset{u=1+4r^2}{=} \int_0^{2\pi} \int_1^5 u^{\frac{1}{2}} \cdot \frac{1}{8}\, du\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} u^{3/2} \Big|_{u=1}^{u=5} d\theta = \int_0^{2\pi} \frac{1}{12}(5^{3/2} - 1)\, d\theta = \boxed{\frac{\pi}{6}(5^{3/2} - 1)}$$

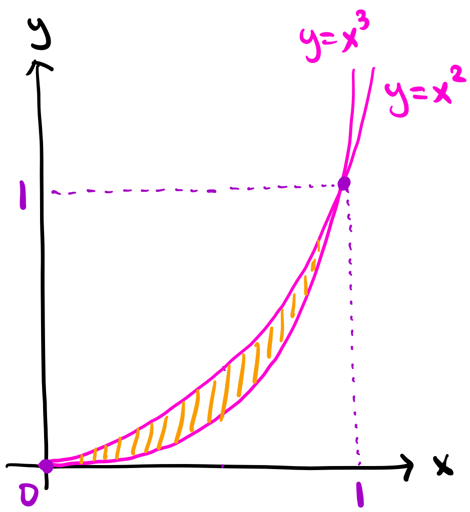
6. The first part asks you to change the order of integration in a double integral and the second part in a triple integral.

a) (5 points) Consider the iterated double integral

$$\int_{x=0}^{1} \int_{y=x^3}^{x^2} f(x,y)\,dy\,dx,$$

which has y first (innermost) and x second. Sketch the region of integration and rewrite the iterated integral so that x is first (innermost) and y is second.

The bounds are $0 \le x \le 1$, $x^3 \le y \le x^2$.



Intersection : $y = x^2$ and $y = x^3$

$\Rightarrow x^2 = x^3 \Rightarrow x = 0, 1$

$\Rightarrow (x,y) = (0,0), (1,1)$.

The boundary curves are

$y = x^2$ and $y = x^3 \rightsquigarrow x = y^{1/2}$ and $x = y^{1/3}$.

$x \ge 0$

The bounds are $0 \le y \le 1$, $y^{1/2} \le x \le y^{1/3}$

$\Rightarrow$ The given integral is equal to $\boxed{\int_0^1 \int_{y^{1/2}}^{y^{1/3}} f(x,y)\,dx\,dy}$

<u>Note</u> You can also algebraically find the bounds.

The outer integral is with respect to dy

$\Rightarrow 0 \le x^3 \le y \le x^2 \le 1 \Rightarrow 0 \le y \le 1$.

$x \ge 0$ $\qquad$ $x \le 1$

$x^3 \le y \Rightarrow x \le y^{1/3}$, $\quad y \le x^2 \Rightarrow y^{1/2} \le x$

$\Rightarrow y^{1/2} \le x \le y^{1/3}$

b) (5 points) Consider the triple integral

$$\int_{z=0}^{1}\int_{y=z}^{1}\int_{x=z}^{1} f(x,y,z)\,dx\,dy\,dz,$$

which has x first (innermost), y second, and z third. Rewrite the iterated integral in reverse order: z first (innermost) , y second, and x third.

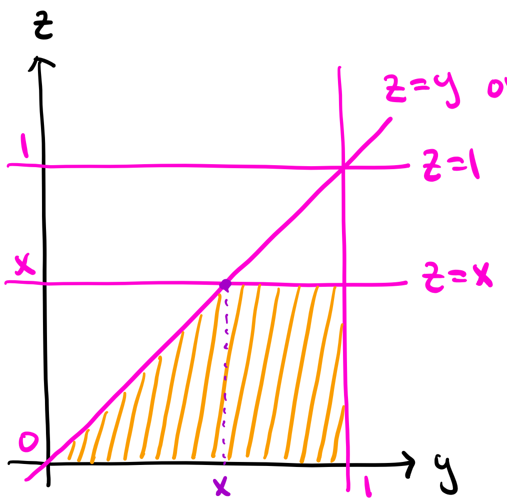
The bounds are $0 \le z \le 1$, $z \le y \le 1$, $z \le x \le 1$.

The outermost integral is with respect to dx

$\Rightarrow 0 \le z \le x \le 1 \Rightarrow 0 \le x \le 1$.

For the inner double integral, we look at the cross section with constant x.

The boundary conditions are $0 \le z \le 1$, $z \le y \le 1$, $z \le x$.



Intersection : $z=y$ and $z=x$

$\Rightarrow (y,z) = (x,x)$.

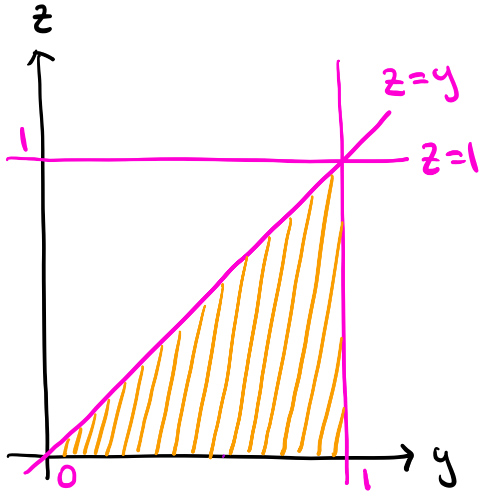
$$\begin{cases} 0 \le y \le x \Rightarrow 0 \le z \le y \\ x \le y \le 1 \Rightarrow 0 \le z \le x. \end{cases}$$

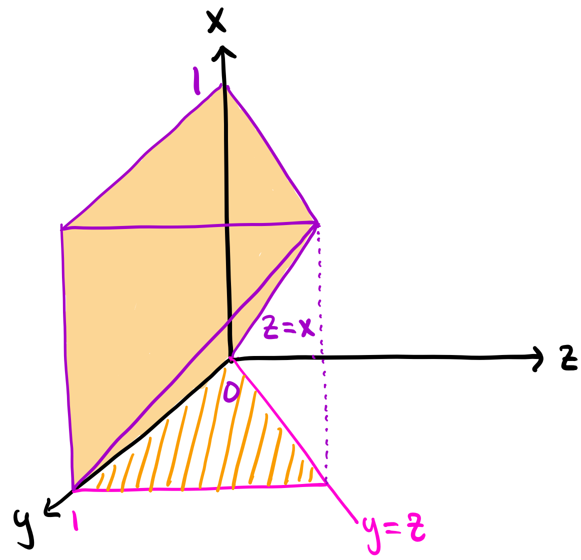
$\Rightarrow$ The given integral is equal to

$$\boxed{\int_0^1\int_0^x\int_0^y f(x,y,z)\,dz\,dy\,dx + \int_0^1\int_x^1\int_0^x f(x,y,z)\,dz\,dy\,dx}$$

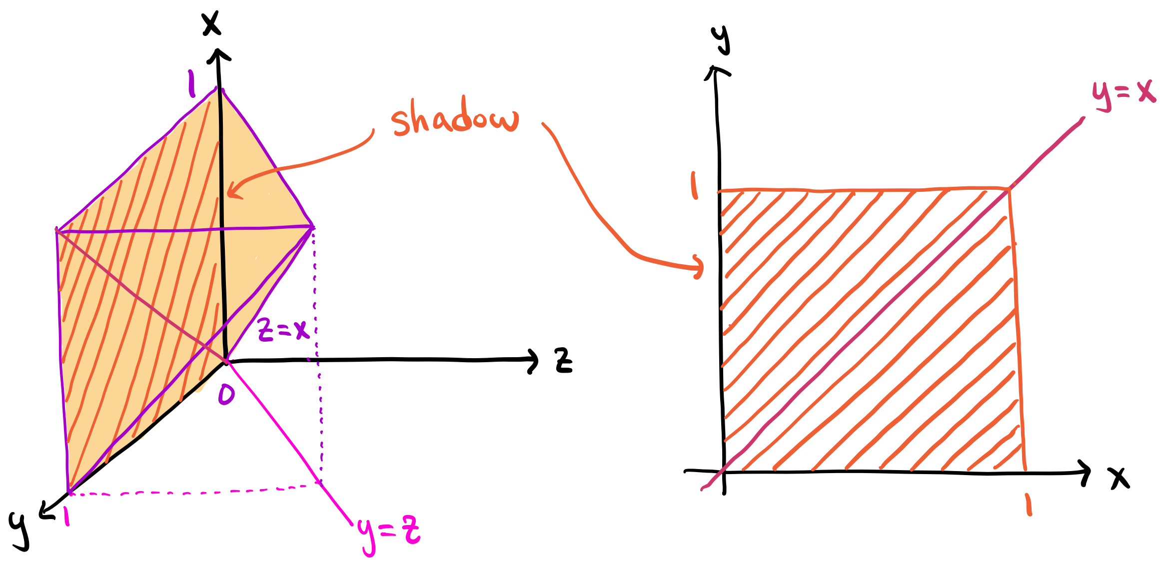
__Note__ You can also use the projection method.

The bounds are $0 \le z \le 1$, $z \le y \le 1$, $z \le x \le 1$.





For the outer double integral, we look at the shadow on the xy-plane.



However, the bounds for z are not uniform:

$$\begin{cases} 0 \le x \le 1,\ 0 \le y \le x \Rightarrow 0 \le z \le y. \\ 0 \le x \le 1,\ x \le y \le 1 \Rightarrow 0 \le z \le x \end{cases}$$

$\Rightarrow$ The given integral is equal to

$$\int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx + \int_0^1 \int_x^1 \int_0^x f(x,y,z)\,dz\,dy\,dx$$

* This problem is not correctly formulated.

7. The first part is a maximization problem and the second part a minimization problem.

a) (5 points) Maximize the product xyz subject to the requirement $2x + y + 3z = 1$.

Set $f(x,y,z) = xyz$, $g(x,y,z) = 2x+y+3z-1$.

Solve $\nabla f = \lambda \nabla g$ and $g=0$

$\Rightarrow (yz, zx, xy) = \lambda(2,1,3)$ and $2x+y+3z-1=0$

$$\Rightarrow \begin{cases} yz = 2\lambda \rightsquigarrow xyz = 2\lambda x \\ zx = \lambda \rightsquigarrow xyz = \lambda y \\ xy = 3\lambda \rightsquigarrow xyz = 3\lambda z \end{cases}$$

$\Rightarrow 2\lambda x = \lambda y = 3\lambda z$

Case 1 $\lambda = 0$: $yz = 2\lambda = 0$, $zx = \lambda = 0$, $xy = 3\lambda = 0$

$\Rightarrow x = y = z = 0 \Rightarrow 2x+y+3z-1 \neq 0$.

$\Rightarrow$ no solutions.

Case 2 $\lambda \neq 0$: $2x = y = 3z$

$2x+y+3z-1 = 0 \rightsquigarrow y+y+y-1 = 0 \rightsquigarrow y = \frac{1}{3}$.

$\Rightarrow x = \frac{1}{6}, y = \frac{1}{3}, z = \frac{1}{9}$.

The maximum value is $f(\frac{1}{6}, \frac{1}{3}, \frac{1}{9}) = \boxed{\frac{1}{162}}$

Note The problem should specify that x, y, z are positive, since otherwise there is no maximum values.

b) (5 points) Find the point on the paraboloid $z = x^2 + y^2$ that is closest to the point $P = (6, 3, 4)$.

Distance from P is $\sqrt{(x-6)^2+(y-3)^2+(z-4)^2}$

We find the minimum of $(x-6)^2+(y-3)^2+(z-4)^2$ subject to the constraint $z=x^2+y^2$.

Set $f(x,y,z)=(x-6)^2+(y-3)^2+(z-4)^2$ and $g(x,y,z)=x^2+y^2-z$.

Solve $\nabla f=\lambda\nabla g$ and $g=0$.

$\Rightarrow (2(x-6),\ 2(y-3),\ 2(z-4)) = \lambda(2x, 2y, -1)$ and $x^2+y^2-z=0$.

$\rightsquigarrow 2(x-6)=2\lambda x,\ 2(y-3)=2\lambda y,\ 2(z-4)=-\lambda,\ x^2+y^2=z$

$\rightsquigarrow x-6=\lambda x,\ y-3=\lambda y,\ 2z-8=-\lambda,\ x^2+y^2=z$.

$\rightsquigarrow (1-\lambda)x=6,\ (1-\lambda)y=3,\ 2z-8=-\lambda,\ x^2+y^2=z$.

$\rightsquigarrow x=\dfrac{6}{1-\lambda},\ y=\dfrac{3}{1-\lambda},\ z=\dfrac{8-\lambda}{2},\ x^2+y^2=z$.

$x^2+y^2=z \rightsquigarrow \left(\dfrac{6}{1-\lambda}\right)^2+\left(\dfrac{3}{1-\lambda}\right)^2=\dfrac{8-\lambda}{2} \rightsquigarrow 90=(1-\lambda)^2(8-\lambda)$

$\rightsquigarrow \lambda^3-10\lambda^2+17\lambda+82=0$

$\rightsquigarrow (\lambda+2)(\lambda^2-12\lambda+41)=0$

$\rightsquigarrow \lambda=-2$

$\Rightarrow x=2,\ y=1,\ z=5$.

The closest point is $\boxed{(2,1,5)}$